Math 2050, HW 5

Q1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $S = \{x \in \mathbb{R} : f(x) = 0\}$. Show that S is closed in the sense that if $x_n \in S$ and $x_n \to x$, then $x \in S$.

Q2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous functions such that

$$f(m2^{-n}) = m2^{-n}$$

for all $m \in \mathbb{Z}, n \in \mathbb{N}$. Show that $f(x) = x$ for all $x \in \mathbb{R}$.

Q3. Let $I = [0, \pi/2]$ and $f : I \to \mathbb{R}$ be a function given by $f(x) = \sup\{x^2, \cos x\}$ for $x \in I$. Show that there is $x_0 \in I$ such that $f(x_0) = \min\{f(x) : x \in I\}$. Moreover, $x_0^2 = \cos x_0$.

Q4. Show that $f(x) = x^{-1}$ on $(a, +\infty)$ is uniformly continuous if $a > 0$. Is the result still true if $a = 0$? Give your reasoning.

Q5. Suppose $f : [0, 1] \to \mathbb{R}$ is a function such that for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \le \Lambda |x - y|^{1/2}$$

for some $\Lambda > 0$. Show that f is uniformly continuous. Is the converse also true? Give your reasoning.